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THE IN- AND-CIRCUMSCRIBED QUADRILATERAL

By W. E. Byerly

In 1792 Fuss[*] showed that if the radii r and R of two circles and the distance a between their centres satisfy the condition

$$[(R+a)^2-r^2][(R-a)^2-r^2]=r^4, \qquad [I]$$

a quadrilateral can be drawn that will be inscribed in the latter and circumscribed about the former.

Six years later Fuss[†] carried his investigation further and obtained the conditions that certain polygons of more than four sides could be inscribed in one of two circles and circumscribed about the other.

In 1822 Poncelet in his *Proprietés Projectives* proved a theorem that greatly increased the value of Fuss's results:—"If a polygon of n sides be inscribed in one conic and circumscribed about another an infinite number of polygons of n sides can be inscribed in the first conic and circumscribed about the second, and moreover if from any point on the circumscribed conic a tangent be drawn to the inscribed conic, from the point where this tangent again meets the circumscribed conic a second tangent be drawn, and this process be continued, the nth tangent will intersect the first on the circumscribed conic and a closed polygon of n sides will be formed."

In 1860 Mention[‡] took up the problem of the polygon of n sides inscribed in a circle and circumscribed about a second circle and solved it for values of n from 3 to 11, giving as the condition for the case $n=4$

$$1-\frac{(R+r+a)(R+r-a)(R-r+a)(R-r-a)}{r^4}=0. \qquad [II]$$

The problem of the In- and-Circumscribed Polygon, or as he calls it the Porism of the In- and-Circumscribed Polygon seems to have greatly interested Cayley who wrote several papers[§] on the subject taking up the problem for two conics and dealing with two circles as a special case. In one of these papers* he parallels Mention's work for values of n from 3 to 9.

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[*] *Nova Acta Petrop.* 1792.

[†] *Nova Acta Petrop.* 1798.

[‡] *Bulletin de l'Acad. de St. Petersburg.* 1860.

[§] Cayley's *Collected Works*, vols. I, II, and IV, passim.

For the quadrilateral Cayley's condition is

$$\frac{R^2-a^2}{r^2}\left[\frac{R^4+r^4+a^4-2a^2r^2-2a^2R^2-2r^2R^2}{r^4}-1\right]=0. \qquad \text{[III]}$$

Fuss's condition [I] and Mention's condition [II] both reduce to

$$R^4+a^4-2a^2r^2-2a^2R^2-2r^2R^2=0 \qquad \text{[IV]}$$

and are identical. Cayley's condition [III] reduces to

$$(R^2-a^2)\ [R^4+a^4-2a^2r^2-2a^2R^2-2r^2R^2]=0 \qquad \text{[V]}$$

and includes the older ones together with the new condition $R^2 - a^2 = 0$.

In the papers to which I have referred, the quadrilateral problem has been treated incidentally and without detail. I wish to consider it independently and from a decidedly elementary point of view.

Let the equations of the inscribed and circumscribed circles be respectively

$$x^2+y^2=r^2, \qquad (1)$$

$$(x-a)^2+y^2=R^2, \qquad (2)$$

and suppose that a quadrilateral can be inscribed in (2) and circumscribed about (1).

Then by Poncelet's Theorem we can start at any point of (2) and draw tangents to (1), from the points where these tangents meet (2) a second time we can draw tangents to (1) and they will intersect on (2); and we shall have a quadrilateral inscribed in (2) and circumscribed about (1).

If we take as our starting point $(a+R,\ 0)$ and draw our quadrilateral as indicated above, it is evident from considerations of symmetry that the last vertex formed will lie on the X axis and therefore that it must be one of the two points in which (2) meets the X axis, i. e. $(a-R,\ 0)$ or $(a+R,\ 0)$.

Case I. Suppose the last vertex is at $(a-R,\ 0)$. Then each of the tangents from $(a+R,\ 0)$ will form with a tangent from $(a-R,\ 0)$ an angle inscribed in a semi-circle and therefore a right angle.

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* Cayley's *Collected Works*, vol. IV, p. 292, et seq.

In Figures 1 and 2 the right triangles OTC and $OT'B$ are equal. Hence $OC = |a - R|$.

In the right triangle AOC the legs are $a + R$ and $|a - R|$ and the distance from the vertex of the right angle to the hypothenuse is r.

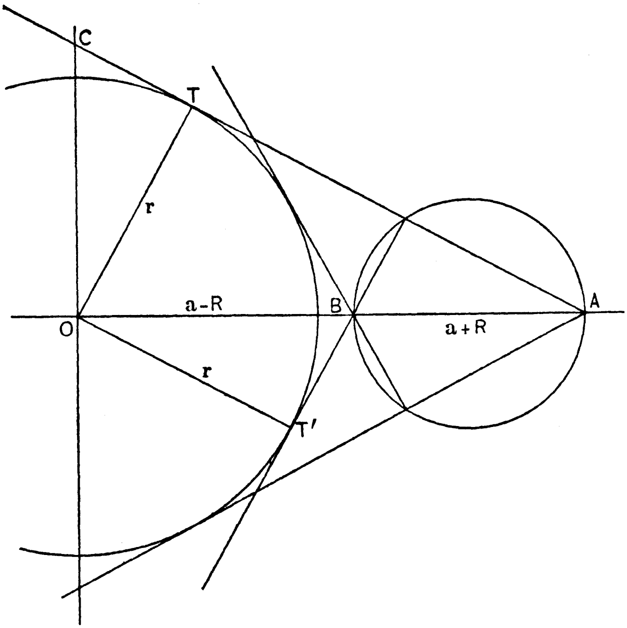

FIG. 1.

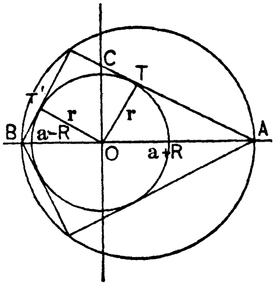

FIG. 2.

Then
$$r = \frac{(a + R)(\,|a - R|\,)}{\sqrt{(a + R)^2 + (a - R)^2}},$$

or
$$\frac{1}{r^2} = \frac{1}{(a + R)^2} + \frac{1}{(a - R)^2}. \qquad [\mathrm{VI}]$$

This condition [VI], which is equivalent to [I] and [II], the conditions obtained by Fuss and Mention, is therefore a necessary condition. That it is sufficient is easily shown as follows:—In figures 1 and 2 through A draw the tangent AT to the circle (1) and from B drop a perpendicular upon it. Assume [VI]; then $OC = |a - R|$, OTC and $OT'B$ are equal triangles,

$OT' = r$ and therefore BT' is tangent to the first circle. Condition [VI] then is necessary and sufficient.

Case II. Suppose the last vertex is at $(a - R, 0)$.

The second pair of tangents coincide with the first and the quadrilateral is of course a degenerate one.

In this case the points of contact of the tangents from $(a + R, 0)$ must lie on the first circle, perpendiculars to these tangents at their points of contact must pass through $(a - R, 0)$ and also through the centre of the first circle. R must then be equal to a or to $-a$, i. e.,

$$R^2 - a^2 = 0. \qquad [\text{VII}]$$

That [VII] is sufficient as well as necessary may be shown as follows:— Let $R = a$. From A, (a, a), draw a tangent to the first circle and drop a perpendicular upon it from B, $(a, -a)$. Join with O the intersection P of these perpendiculars, which of course is a point on the second circle.

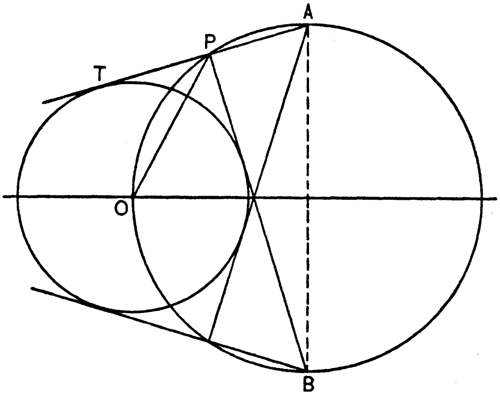

FIG. 3.

The angle OPB, being measured by one half the quadrant OB, is an angle of 45 degrees. OP bisects the angle BPT, and O is equidistant from AT and BP. Therefore BP is a tangent and condition [VII] is sufficient as well as necessary.

The complete condition covering Case I and Case II is, then, [III] the condition given by Cayley.

The following construction by which an inscribed circle corresponding to any given circumscribed circle may be found is interesting.

Join any point P on the circumference of the given circle (Fig. 4) with the extremities A and B of a horizontal diameter and with the extremities C and D of a vertical diameter. Taking as a centre any one of the four points O', O'', C, D, where the second pair of lines, which are easily seen to bisect the angles between the first pair, intersect the horizontal and vertical diameters, with its distance from either of the first pair as a radius, describe a circle and it will be the circle required.

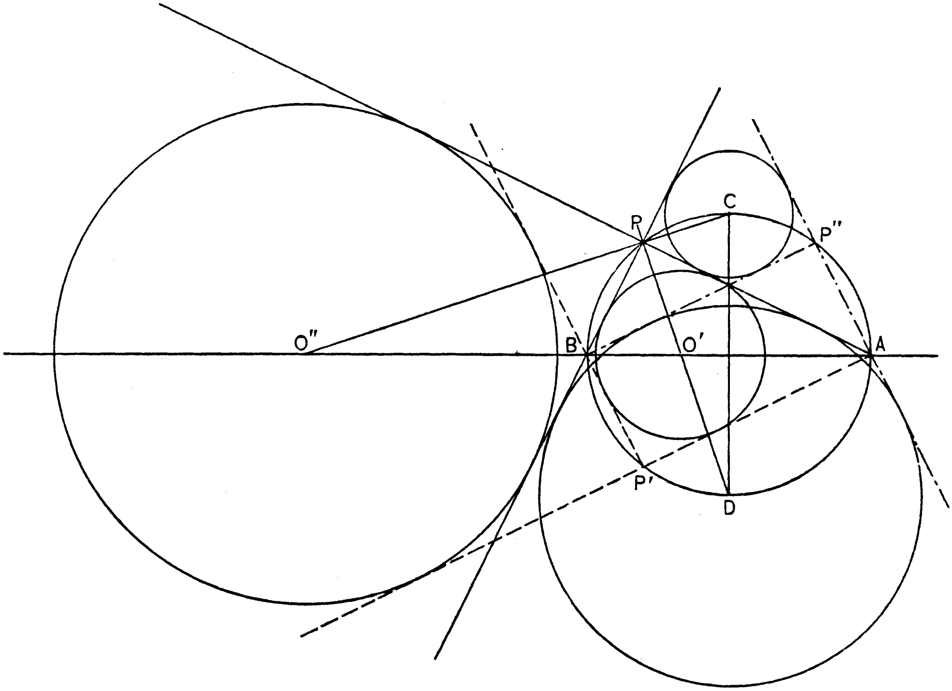

FIG. 4.

The first pair of lines together with their reflections in the horizontal diameter will form a quadrilateral $APBP'$ inscribed in the given circle and circumscribed about the two circles whose centres O' and O'' are in the horizontal diameter.

The first pair of lines together with their reflections in the vertical diameter will form a quadrilateral $APBP''$ inscribed in the given circle and circumscribed about the two circles whose centres C and D are on the vertical diameter.

The first pair of circles, one of which is wholly within and the other wholly without the given circle, come under Case I and satisfy condition [VI].

The second pair of circles, each of which lies partly within and partly without the given circle, come, under Case [II] and satisfy condition [VII].

Every circle that can be combined with the given circle to form an in- and circumscribed pair can be obtained by the construction just given, followed if necessary by a rotation through a suitable angle about the centre of the given circle.

Harvard University,
Cambridge, Mass.